Annular Borda flow

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A remarkably stable type of revolving hollow water jet was generated in a vertical pipe, fixed in the base of a large tank, by first establishing Borda free flow, in which the jet springs clear of the pipe wall. Swirl was then imparted to the oncoming water, and an air core formed in the jet which was of varicose shape with alternate swellings and contractions of both its inner and outer surfaces. The observed wavelengths are compared with the theory, in which inertia and surface tension but not viscosity and gravity are taken into account.

1. Introduction

Hollow swirling water jets, in which the surface tension and centrifugal forces balance, were examined by Binnie & Davidson (1949). The existence of this type of jet was discovered accidentally in the course of experiments on swirling flow under gravity through a sharp-edged orifice in the bottom of a tank. When emptying of the tank was almost completed, the jet was sometimes seen to contain a narrow air core persisting for some distance. The jet showed variations in diameter akin to those in the vibrating 'solid' jet investigated by Rayleigh (1878, 1879). A theory of the annular jet was developed on the same lines that Lamb (1932) employed in his presentation of Rayleigh's work.

In those experiments the stream emerging from the tank was to some extent disturbed by the boundary layer on the floor of the tank, and it was more recently thought that better results might be achieved if, instead of the orifice plate, a vertical re-entrant mouthpiece were used, in which Borda free flow could be established. In this kind of motion the jet springs clear from the top of the pipe. Boundary-layer formation is slight, and the jet is the most stable that it is possible to create. After the jet had been started, a small amount of rotation could be imparted to the oncoming water in order to reach the desired conditions. An account of experiments on these lines is given below. A considerable measure of success was achieved, as may be seen in figure 1, plate 1, and much longer annular jets were obtained than were possible in the earlier work.

2. Description of the apparatus

The experiments were conducted in an open circular tank of 3 ft. depth and 5 ft. diameter. A purely radial water supply entered through a casting bolted externally over a central hole in the bottom of the tank. Thence it passed into the tank and was directed outwards towards the periphery by a horizontal baffle plate. A drawing of the arrangement was given by Binnie & Hookings

(1948). The supply with swirl was provided by two tangential pipes fixed at opposite ends of a diameter. The re-entrant glass pipe, of 1·34 in. outside diameter and with a wall thickness about 0·1 in., was 41 in. long, and it was fixed centrally to the casting so that a length of 26 in. protruded above the baffle plate. Attempts to improve its lip by grinding were unsuccessful as it proved impossible to avoid fragmenting the edges, therefore a brass cap was fitted, of 1·45 in. outside diameter and 1·18 in. inside diameter. To ensure that no appreciable vacuum could form inside the pipe and distend the core, a short stub (see figure 1, plate 1) was attached 3 in. below the lip and connected to the atmosphere. This precaution turned out to be worth while, as a vacuum of about 1/20 in. of water was measured when the connecting tube was led to a manometer; this pressure was of the same order of magnitude as the surface-tension forces. The photographs of the jet and the glass pipe were obtained by means of a submerged mirror inclined at 45° to the horizontal, the camera being mounted above the tank and pointed vertically downwards. An intense source of light was placed above the water and behind the pipe.

3. Description of the experiments

To block the top of the pipe when necessary, a rubber bung was employed with an open tube of small diameter as a handle; this could stop the motion without shock, for air was drawn down through the handle as the flow ceased. The first step in a sequence of experiments was to insert the bung and fill up the tank with the radial supply until the water level stood a couple of inches above the lip. After an interval long enough for eddies to die down, the bung was cautiously raised, whereupon Borda free flow was often but by no means always produced. A typical jet obtained in this way is shown in figure 1 (*a*) with a scale graduated in tenths of an inch alongside. It can be seen inside the glass tube, diminishing in diameter as its velocity was increased by gravity. A minute dimple could just be discerned in the free surface at the top of the original photograph; therefore in this instance the flow was not entirely free from swirl. Some irregularities are visible in the upper part of the profile of the jet. Those close to the top were probably real, being caused by slight imperfections in the lip; this is an effect often seen in the flow over straight weirs. Lower down there are some high lights that may have originated in inaccessible water drops clinging to the inner surface of the pipe. The jet remained smooth and glassy not only through the rest of the pipe but also through a further distance of $3\frac{1}{2}$ ft., as far as an intercepting tank into which the jet fell. Indeed, after some minutes the flow was so quiet and steady that a casual observer might be unaware that motion was in progress.

To form an annular jet the tangential supply valve was slightly opened, and satisfactory results were produced if the radial supply were adjusted to maintain the water level about $1\frac{3}{4}$ in. above the lip. As the swirl developed, a dimple appeared in the surface, and this extended further and further downwards with a tail of increasing sharpness that threw off minute bubbles into the pipe. Then, without an intermediate stage, a core of very small diameter could be seen to extend throughout the visible length of the pipe. With more swirl the core increased to the size shown in figure 1 (*b*), and on it waves fixed in space are just visible. In the pipe the jet remained glassy, although in this example the core near the bottom of the photograph was somewhat eccentric. Below the tank the jet was disturbed, though not sufficiently for drops to be ejected from it. In these experiments with swirl the projecting vertical scale shown in figure 1 (*a*) had to be removed, and it was replaced by a narrow transparent scale held at the back of the pipe. Some of the numerals at 1 in. intervals can be discerned.

With still more swirl figures 1 (*c*) and (*d*) were obtained, showing undulations that are easily seen. Again the jet throughout the pipe was glassy but much disturbed below it. Occasionally a loud note was emitted from the intercepting tank, suggesting that travelling waves had formed on the lowest part of the jet, but it persisted for too short a time for its frequency to be measured. The core in figure 1 (*d*) was the largest that was produced, and it burst soon after the photograph was taken.

4. Theory of the waves

As in the analysis put forward by Binnie & Davidson (1949), the simplifying assumptions are made that the liquid is inviscid and the motion irrotational, also that gravity is inoperative and the amplitude of the undulations is infinitesimal. Binnie & Davidson's theory was worked out for waves of maximum instability, which are stationary relative to the mean axial motion of the fluid, but we are now concerned with stable waves that in practice are fixed in space so that their phase velocity relative to the fluid is equal in magnitude to the jet velocity. By taking a reference frame moving with the fluid, we may consider a jet possessing tangential but no axial velocity, and our aims are to find the corresponding theoretical velocity of travelling waves and to compare it with the measured jet velocity.

First, the equilibrium of the undisturbed jet must be examined. Since the motion is taken to be irrotational, the tangential velocity v at radius r is given by $v = S/r$, where S is a constant; the inner and outer radii are denoted by a and b. The equation of radial equilibrium shows that, if ρ is the density, the distribution of pressure p is given by

$$p = \text{const.} - \tfrac{1}{2}\rho S^2/r^2. \tag{1}$$

The surface conditions are

$$p = \gamma/a \quad \text{at} \quad r = a, \quad \text{and} \quad p = -\gamma/b \quad \text{at} \quad r = b, \tag{2}$$

where γ is the surface tension. The insertion of (2) into (1) then leads to

$$(a-b)/ab = 2\gamma'/S^2, \tag{3}$$

where γ' is the kinematic surface tension.

To satisfy Laplace's equation in cylindrical co-ordinates (r, θ, z), the appropriate form of the velocity potential representing varicose disturbances is

$$\phi = \{AI_0(kr) + BK_0(kr)\}\cos(kz - \sigma t) - S\theta, \tag{4}$$

where t denotes time, and A and B are constants to be determined from the pressure conditions at the inner and outer free surfaces. We take the equation of the outer surface to be

$$r = a + \xi, \tag{5}$$

where ξ is the (infinitesimal) displacement due to the wave. Since

$$\partial\xi/\partial t = \partial\phi/\partial r \quad \text{at} \quad r = a, \tag{6}$$

it follows that

$$\xi = -k\sigma^{-1}\{AI_1(ka) - BK_1(ka)\}\cos(kz - \sigma t). \tag{7}$$

It was shown by Lamb (1932, §274) that the pressure at the outer surface is

$$p = \gamma\left\{\frac{1}{a} - \frac{\xi}{a^2} - \frac{\partial^2\xi}{\partial z^2}\right\}. \tag{8}$$

We also have from Bernoulli's equation that this pressure is given by

$$\begin{aligned}\frac{p}{\rho} &= -\frac{\partial\phi}{\partial t} - \frac{1}{2}\left(\frac{S}{a+\xi}\right)^2 + \text{const.}\\ &= -[\sigma\{AI_0(ka) + BK_0(ka)\} + (S^2k/a^3\sigma)\{AI_1(ka) - BK_1(ka)\}]\cos(kz - \sigma t)\\ &\qquad - \tfrac{1}{2}S^2/a^2 + \text{const.}\end{aligned} \tag{9}$$

Here a term, previously overlooked, has been inserted, thanks to the vigilance of Dr Brooke Benjamin. When (8) and (9) are equated, after (7) has been substituted in (8), the terms invariable with time cancel in virtue of (1) and (2); and after use has been made of (3) the remainder yield

$$\sigma^2 = \gamma'\frac{k}{a^2}\left(k^2a^2 - 1 - \frac{2b}{a-b}\right)\left\{\frac{I_1(ka) - xK_1(ka)}{I_0(ka) + xK_0(ka)}\right\}, \tag{10}$$

where $x = B/A$.

The motion at the inner surface, denoted by

$$r = b + \eta, \tag{11}$$

is dealt with in the same way; and corresponding to (7) it is found that

$$\eta = -k\sigma^{-1}\{AI_1(kb) - BK_1(kb)\}\cos(kz - \sigma t). \tag{12}$$

Consequently

$$\sigma^2 = \gamma'\frac{k}{b^2}\left(1 - k^2b^2 - \frac{2a}{a-b}\right)\left\{\frac{I_1(kb) - xK_1(kb)}{I_0(kb) + xK_0(kb)}\right\}. \tag{13}$$

Equations (10) and (13) are to be taken together. For a jet of given radii a and b, we choose a value of k and calculate first x and then σ. Thus the wave velocity σ/k can be found for the wavelength $\lambda = 2\pi/k$.

5. Comparison between the theoretical and the observed wavelengths

To determine the outer and inner diameters of the jets by measurement off the photographs to the same scale as the outer diameter of the pipe, it is necessary to determine the refraction effects of the glass, of the air between it and the jet, and of the annular thickness of the jet. An estimate indicates that the apparent outer diameter of the jet should be multiplied by 1·33 and that no correction should be applied to the inner diameter. A rough check on the factor 1·33 was made by measuring the jet diameter and the pipe outer diameter on a print of figure 1(a) close to the cap. These dimensions were 0·34 and 0·84 in. As the true value of the second length was 1·34 in., the corrected jet diameter was $1 \cdot 33 \times 1 \cdot 34 \times 0 \cdot 43/0 \cdot 84 = 0 \cdot 91$ in. The inner diameter of the cap was 1·18 in., and this leads to a coefficient of contraction 0·77 based on diameter; the well-known approximate theory yields $2^{-\frac{1}{2}} = 0 \cdot 71$. The corrected mean outer and inner radii of the jets, measured in the regions immediately below the venae contractae, are shown in the table below, to which the values of S calculated from (3) have been added, with γ' taken as 4·5 in.^{3}/sec^2.

Figure	Outer radius (a in.)	Inner radius (b in.)	S (in.^{2}/sec)
1 (b)	0·41	0·04	0·63
1 (c)	0·40	0·08	0·95
1 (d)	0·41	0·115	1·20

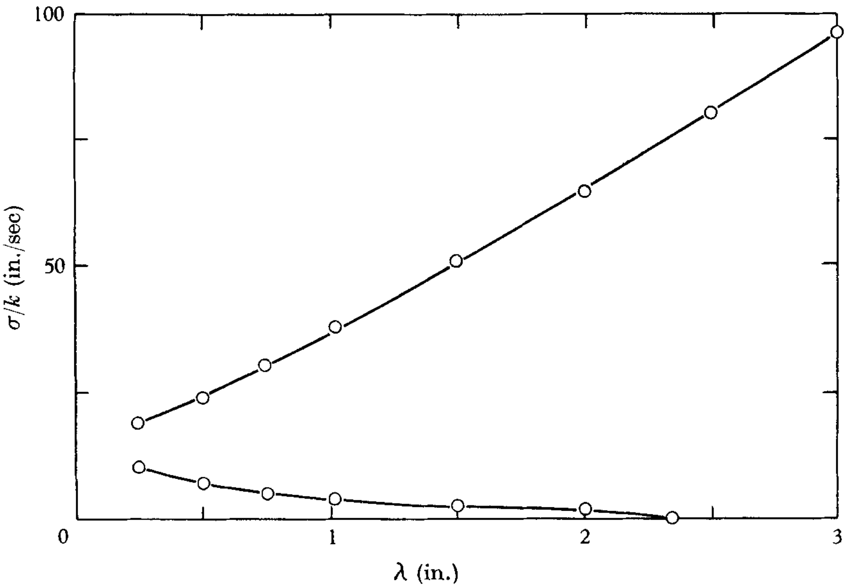


FIGURE 2. Theoretical variation of the wave velocity σ/k with the wavelength λ for the conditions of figure 1 (b).

A numerical exploration of the theoretical results was made for the radii in figure 1 (b); the waves there are of amplitude small compared with those in figure 1 (c) and (d). For 8 values of λ in the range $\frac{1}{4}$ to 3 in., x and σ/k were calculated from (10) and (13), and the results are plotted in figure 2. The lower curve, giving velocities of waves too slow to be observable, was derived from positive values of x; and from (7) and (12) these disturbances were found to be opposite in sign. The curve terminates at $\lambda = 2 \cdot 34$ in. where the expression in parentheses in (10) change sign; its counterpart in (13) is negative throughout the range.

The upper curve is for disturbances of the same sign, and it shows velocities of the magnitude employed in the experiments. As the cross-section of the core was usually small compared with the total cross-section and the outer diameter of the jet diminished with distance travelled, it is not quite obvious from the photographs whether the outer and inner surfaces were in phase or antiphase. However, the former type is seen if a close examination of figure 1 (d) is made.

Figures 1 (*c*) and (*d*) clearly show wavelengths increasing with velocity, and this is in accordance with the trend of the upper curve.

Measurements on a large print of figure 1 (*b*) showed that the half wave below the uppermost swelling occupies a length of about $\frac{3}{4}$ in.; and for $\lambda = 1\frac{1}{2}$ in. figure 2 yields 50 in./sec. The discharge was 28 in.^{3}/sec, and the jet velocity as calculated from the cross-section was 54 in./sec. These measurements were very approximate; nevertheless, they suggest that the distortion due to gravity was not very great. Even if a theory had been available that covered the effects of gravity, it would not have wholly met the experimental conditions. Viscosity may be important near the core; and Binnie & Teare (1956) showed that, in swirling flow passing under gravity through a trumpet entrance to a vertical pipe, a boundary layer forms on the core and is ejected downwards at a higher velocity than the mean taken over the cross-section. In the present circumstances a similar effect may be expected close to the outer surface also.

REFERENCES

Binnie, A. M. & Davidson, J. F. 1949 *Proc. Roy. Soc.* A, **199**, 443.

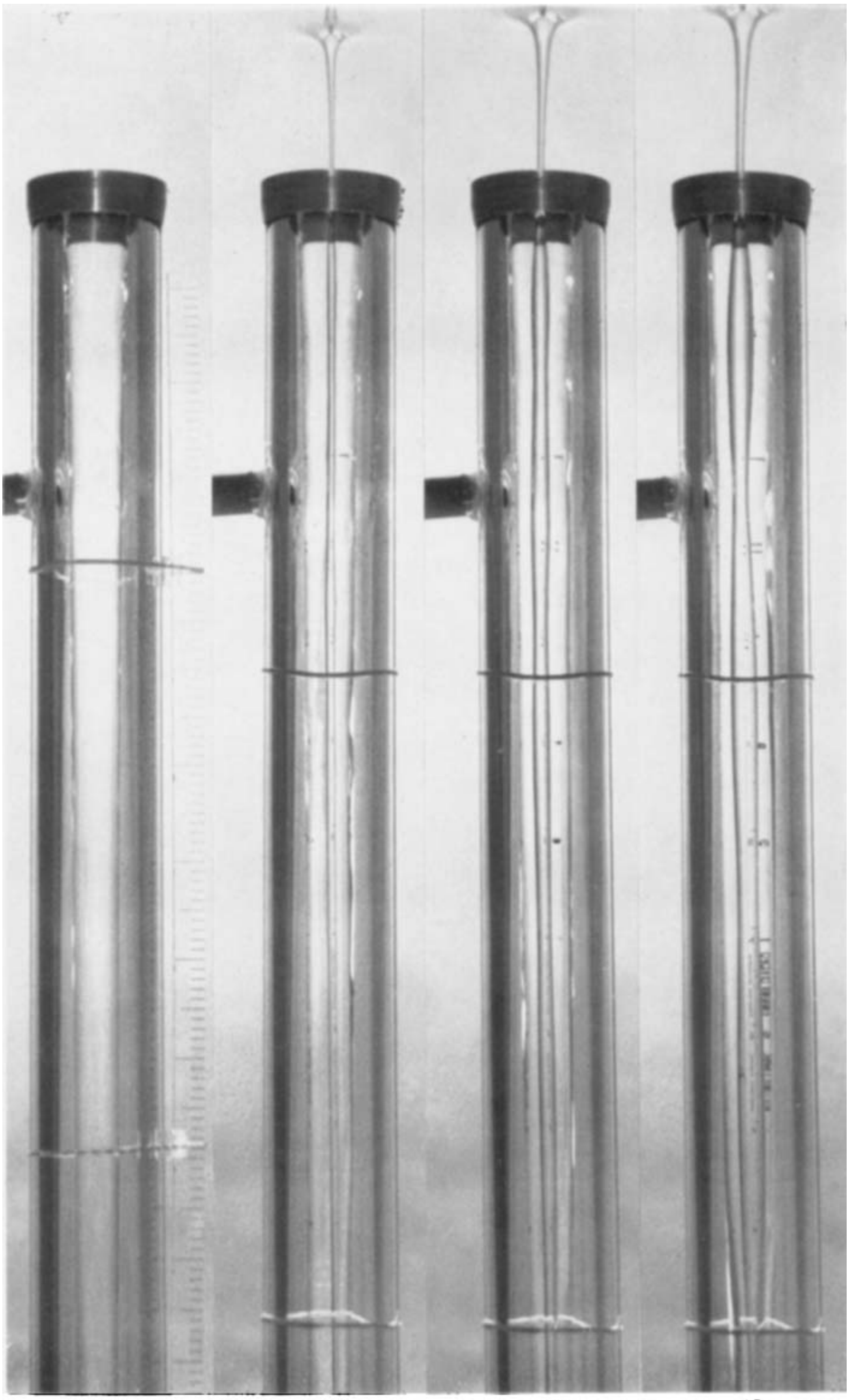
Binnie, A. M. & Hookings, G. A. 1948 *Proc. Roy. Soc.* A, **194**, 400.

Binnie, A. M. & Teare, J. D. 1956 *Proc. Roy. Soc.* A, **235**, 78.

Lamb, Sir H. 1932 *Hydrodynamics*, 6th ed. Cambridge University Press.

Rayleigh, Lord 1878 *Proc. Lond. Math. Soc.* **10**, 4 (*Papers* **1**, 361).

Rayleigh, Lord 1879 *Proc. Roy. Soc.* **29**, 71 (*Papers* **1**, 377).



(*a*) (*b*) (*c*) (*d*)

FIGURE 1. (*a*) Borda free jet. (*b*) Small air core formed. (*c*) Larger core, clearly of varicose shape. (*d*) Largest possible core.